Blatt 1 :

Hausaufgaben :

H1

$$\int_0^1 \sqrt{t(1-t)}\,dt = \int_0^1 t^{3/2-1}\cdot(1-t)^{3/2-1}\,dt \overset{14.11}{=} B\left(\tfrac{3}{2},\tfrac{3}{2}\right) = \frac{\Gamma\left(\frac{3}{2}\right)^2}{\Gamma(3)} =$$

$$\underset{\substack{\Gamma(s+1)=\Gamma(s)\cdot s\\ s>0}}{\overset{14.9}{=}} \frac{\left(\Gamma\left(\frac{1}{2}\right)\cdot\frac{1}{2}\right)^2}{2!} \underset{\substack{\Gamma(\frac{1}{2})=\sqrt{\pi}\\ \Gamma(1)=1}}{\overset{14.12}{=}} \frac{\pi\cdot 1/4}{2} = \frac{\pi}{8}$$

$$\int_0^1 \left(t(1-t)\right)^{3/2}dt = \int_0^1 t^{5/2-1}(1-t)^{5/2-1}\,dt = B\left(\tfrac{5}{2},\tfrac{5}{2}\right) = \frac{\Gamma(5/2)^2}{\Gamma(5)} =$$

$$= \frac{\left(\Gamma(1/2)\cdot 3/2\cdot 1/2\right)^2}{4!} = \frac{\pi\cdot 9/16}{2\cdot 3\cdot 4} = \frac{3\pi}{128}$$

H2

a $x, s > 0$.

$$\int \frac{1}{x(\ln x)^s}\,dx \underset{\substack{\text{Subst}\\ \ln x = t\\ \frac{1}{x}dx = dt}}{=} \int \frac{1}{t^s}\,dt = \frac{t^{-s+1}}{-s+1} \overset{\text{Resubst}}{=}$$

$$= \frac{1}{(1-s)(\ln x)^{s-1}}$$

b

$$\int_2^\infty \frac{1}{x\cdot(\ln x)^s}\,dx = \lim_{b\to\infty}\int_2^b \frac{1}{x\cdot(\ln x)^s}\,dx \overset{a}{=}$$

$$= \lim_{b\to\infty}\left(\frac{1}{(1-s)(\ln x)^{s-1}}\Bigg|_2^b\right) = \lim_{b\to\infty}\frac{1}{(1-s)(\ln b)^{s-1}} - \frac{1}{(1-s)(\ln 2)^{s-1}}$$

$$= \begin{cases} \dfrac{1}{(s-1)\cdot(\ln 2)^{s-1}}, & \text{für } s>1 \\ \infty, & 0<s\leq 1 \end{cases}$$

$\Rightarrow$ Existenz für $s > 1$.

(c) $\sum_{n=2}^{\infty} \frac{1}{n \cdot (\ln n)^s}$ konvergiert für $s > 1$, denn:

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^s} = \sum_{n=1}^{\infty} \frac{1}{(n+1)(\ln n+1)^s}$$

Setze $f(x) := \frac{1}{(x+1)\cdot(\ln x+1)^s}$.

Dann $f: [1,\infty) \to [0,\infty)$ mon. fallend (klar)

und $\int_1^{\infty} f(x)\,dx = \int_2^{\infty} \frac{1}{x\cdot(\ln x)^s}\,dx$ exist. für $s>1$ nach (b).

14.6. (Integralkrit.)

$\Longrightarrow$ Beh.

H3

a) $s \in \mathbb{R}$: $\displaystyle\int_1^\infty \frac{\ln x}{x^s}\,dx$

$$\int_1^\infty \frac{\ln x}{x^s}\,dx \underset{\substack{\text{Subst.}\\ y=\ln x\\ dy=\frac{1}{x}dx\\ (e^y=x)}}{=} \int_0^\infty y\cdot e^{(1-s)y}\,dy = \lim_{\beta\to\infty}\int_0^\beta \underset{\downarrow}{y}\cdot \underset{\uparrow}{e^{(1-s)y}}\,dy =$$

$$\overset{PI}{=} \lim_{\beta\to\infty}\left\{ y\cdot\frac{1}{1-s}\cdot e^{(1-s)y}\Big|_0^\beta - \int_0^\beta \frac{1}{1-s}e^{(1-s)y}\,dy\right\} =$$

$$= \lim_{\beta\to\infty}\left\{\frac{\beta\cdot e^{(1-s)\beta}}{1-s} - \frac{1}{(1-s)^2}\cdot e^{(1-s)y}\Big|_0^\beta\right\} =$$

$$= \lim_{\beta\to\infty}\left\{\frac{\beta\cdot e^{(1-s)\beta}}{1-s} - \frac{e^{(1-s)\beta}}{(1-s)^2} + \frac{1}{(1-s)^2}\right\} < \infty \iff \underline{\underline{s>1}}.$$

b) $f:[1,\infty)\to[0,\infty)$, $f(x)=\frac{\ln x}{x^s}$ monoton fallend. $\forall s>0$

Integralkrit. $\Rightarrow \sum_{n=1}^\infty f(n)$ konv $\Leftrightarrow \int_1^\infty f(x)\,dx$ exist.

$\Rightarrow \sum_{n=1}^\infty \frac{\ln n}{n^s}$ konv. $\left(\Rightarrow \sum_{n=2}^\infty \frac{\ln 2}{n^s} \text{ konv.}\right)$

(H4) Stirlingsche Formel: $\frac{n!}{n^n e^{-n} \sqrt{2\pi n}} \xrightarrow{n\to\infty} 1$

(a) $\binom{2n}{n} = \frac{(2n)!}{n!\,n!} \approx \frac{(2n)^{2n} e^{-2n} \sqrt{2\pi \cdot 2n}}{\left(n^n e^{-n} \sqrt{2\pi n}\right)^2} =$

$= \frac{2^{2n} \cdot n^{2n} \cdot e^{-2n} \cdot 2 \cdot \sqrt{\pi n}}{n^{2n} \cdot e^{-2n} \cdot 2 \cdot \pi n} = \frac{2^{2n}}{\sqrt{\pi n}}$

$\Rightarrow \binom{2n}{n} = \mathcal{O}\left(\frac{4^n}{\sqrt{\pi n}}\right)$

(b) $14.10 \Rightarrow \Gamma(x+n) = (x+n-1)(x+n-2) \cdot \ldots \cdot (x+1) \cdot x \cdot \Gamma(x)$, $x > 0$.

$\Rightarrow \frac{\Gamma(x+n)}{\Gamma(y+n)} \cdot n^{y-x} = \frac{(x+n-1)(x+n-2) \ldots (x+1) \cdot x \cdot \Gamma(x)}{(y+n-1) \cdot \ldots \cdot (y+1) \cdot y \cdot \Gamma(y)} \cdot n^{y-x} =$

$= \frac{n!\, n^y}{y(y+1) \cdot \ldots \cdot (y+n-1)(y+n)} \cdot \frac{x \cdot (x+1) \cdot \ldots \cdot (x+n-1)(x+n)}{n!\, n^x} \cdot \frac{\Gamma(x)}{\Gamma(y)} \cdot \frac{y+n}{x+n}$

$\xrightarrow[(14.10)]{n\to\infty} \Gamma(y) \cdot \frac{1}{\Gamma(x)} \cdot \frac{\Gamma(x)}{\Gamma(y)} \cdot \underline{1} = \underline{1}$

H5

a) $k \in [0,1[$, $t \in]0,1[$, es gilt:

$$0 \le \int_0^t \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} \le \frac{1}{\sqrt{1-k^2t^2}} \int_0^t \frac{dx}{\sqrt{1-x^2}}$$

$$= \frac{1}{\sqrt{1-k^2t^2}} (\arcsin t - \arcsin 0) \xrightarrow[t \uparrow 1]{} \frac{1}{\sqrt{1-k^2}} \cdot \frac{\pi}{2}$$

Also ex. $\int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = \lim_{t \uparrow 1} \int_0^t \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$

nach dem Majorantenkrit.

b) $s > 0$:

$$\int_0^1 \frac{dx}{(1-x^2)^s} = \int_0^{1/2} \frac{dx}{(1-x^2)^s} + \lim_{t \uparrow 1} \int_{1/2}^t \frac{dx}{(1-x^2)^s}$$

Betrachte $0 \le \int_{1/2}^t \frac{dx}{(1-x^2)^s}$ $\quad y = 1-x^2$, $x = \sqrt{1-y}$

$dx = -\frac{1}{2\sqrt{1-y}}\, dy$

$$= \frac{1}{2} \int_{1-t^2}^{3/4} \frac{dy}{y^s \sqrt{1-y}} \le \int_{1-t^2}^{3/4} \frac{dy}{y^s}$$ ex. für $t \uparrow 1$ und $0 < s < 1$.

Für $s \ge 1$ gilt:

$$\int_{1-t^2}^{3/4} \frac{dy}{y^s \sqrt{1-y}} \underset{y \in [0,3/4]}{\ge} \frac{1}{t} \int_{1-t^2}^{3/4} \frac{dy}{y} = \frac{1}{t}\left(\ln \tfrac{3}{4} - \ln(1-t^2)\right) \xrightarrow[t \uparrow 1]{} \infty$$

Das uneigentliche Integral $\int_0^1 \frac{dx}{(1-x^2)^s}$ ex. somit genau dann, wenn $0 < s < 1$.

c) Nach de l'Hospital gilt:

$$\lim_{x\downarrow 0} \frac{\ln(\sin x)}{\ln x} = \lim_{x\downarrow 0} \frac{\frac{1}{\sin x}\cdot \cos x}{\frac{1}{x}} = \lim_{x\downarrow 0} \frac{x\cos x}{\sin x} = 1$$

Weiterhin

$$\int_0^1 \ln x \, dx = \lim_{t\downarrow 0} \int_t^1 \ln x \, dx = \lim_{t\downarrow 0} \left[x\ln x - x \right]_{x=t}^{1}$$

$$= \lim_{t\downarrow 0} (-1 - t\ln t + t) = -1 \text{, denn}$$

$$\lim_{t\downarrow 0} t\cdot \ln t = \lim_{t\downarrow 0} \frac{\ln t}{\frac{1}{t}} \underset{(\text{de l'H.})}{=} \lim_{t\downarrow 0} \frac{\frac{1}{t}}{-\frac{1}{t^2}}$$

$$= \lim_{t\downarrow 0} -t = 0$$

Nach dem Grenzwertkriterium 14.4 ex. somit auch das uneigentliche Integral $\int_0 \ln(\sin x)\, dx$.

d) Für spätere Zwecke zeigen wir vorab, daß für $\alpha > 0$ das uneigentliche Integral $\int_\pi^\infty \frac{\sin x}{x^\alpha} dx$ existiert.

$$z_n := \int_\pi^{n\pi} \frac{\sin x}{x^\alpha} dx = \sum_{k=1}^{n-1} \int_{k\pi}^{(k+1)\pi} \frac{\sin x}{x^\alpha} dx$$

$$= \sum_{k=1}^{n-1} (-1)^k \int_{k\pi}^{(k+1)\pi} \frac{|\sin x|}{x^\alpha} dx =: \sum_{k=1}^{n-1} (-1)^k w_k$$

wobei (w_k) eine monoton fallende Nullfolge ist und somit $\lim_{n\to\infty} z_n$ nach dem Leibnizkriterium existiert. Somit:

$$\int_\pi^\infty \frac{\sin x}{x^\alpha} dx = \lim_{t\uparrow\infty} \int_\pi^t \frac{\sin x}{x^\alpha} dx$$

$$= \lim_{t\uparrow\infty} \int_\pi^{\lfloor \frac{t}{\pi} \rfloor \pi} \frac{\sin x}{x^\alpha} dx + \int_{\lfloor \frac{t}{\pi} \rfloor \pi}^{t} \frac{\sin x}{x^\alpha} dx$$

Dabei konvergiert der linke Term nach obiger Überlegung und für den rechten Term gilt

$$\left| \int_{\lfloor t/\pi \rfloor t}^{t} \frac{\sin x}{x^\alpha}\, dx \right| \le w_{\lfloor t/\pi \rfloor} \xrightarrow[t \uparrow \infty]{} 0 .$$

Also gilt:

$$\int_{\pi}^{\infty} \frac{\sin x}{x^\alpha}\, dx \quad \text{ex.} \quad \forall \alpha > 0. \qquad (*)$$

Jetzt:

$$\int_0^\infty \sin(x^2)\, dx = \int_0^{\sqrt{\pi}} \sin(x^2)\, dx + \lim_{t \uparrow \infty} \int_{\sqrt{\pi}}^{t} \sin(x^2)\, dx$$

Substituiere bei letzterem Integral:

$$y = x^2, \quad x = \sqrt{y}, \quad dx = \frac{1}{2\sqrt{y}}\, dy$$

$$\int_{\sqrt{\pi}}^{t} \sin(x^2)\, dx = \frac{1}{2} \int_{\pi}^{t^2} \frac{\sin y}{y^{1/2}}\, dy \xrightarrow[t \uparrow \infty]{(*)} \frac{1}{2} \int_0^\infty \frac{\sin x}{x^{1/2}}\, dx$$

Damit ex. das uneigentliche Integral $\int_0^\infty \sin(x^2)\, dx$.

e) $$\left| \int_0^1 \frac{dx}{\ln x} \right| = \int_0^1 \frac{1}{-\ln x}\, dx \ge \int_{1/2}^1 \frac{1}{-\ln x}\, dx$$

$$= \lim_{t \uparrow 1} \int_{1/2}^{t} \frac{1}{-\ln x}\, dx \qquad , \quad x = e^y, \quad dx = e^y\, dy$$

$$= \lim_{t \uparrow 1} \int_{\ln 1/2}^{\ln t} \frac{e^y}{-y}\, dy \ge \frac{1}{2} \lim_{t \uparrow 1} \int_{\ln 1/2}^{\ln t} \frac{1}{-y}\, dy = \infty$$

Daher ex. das uneigentliche Integral $\int_0^1 \frac{dx}{\ln x}$ nicht!

(f) $\int_0^1 \sin(\frac{1}{x})dx = \int_{1/\pi}^{1} \sin 1/x \, dx + \lim_{t \downarrow 0} \int_t^{1/\pi} \sin 1/x \, dx$

Subst. bei letztem Integral:

$y = \frac{1}{x}, \; x = \frac{1}{y}, \; dx = -\frac{1}{y^2}dy$

$$\int_t^{1/\pi} \sin 1/x \, dx = \int_\pi^{1/t} \frac{\sin y}{y^2} dy \xrightarrow[t \downarrow 0]{\circledast \text{ aus } (d)} \int_\pi^\infty \frac{\sin x}{x^2} dx$$

Damit exist. das uneigentliche Integral $\int_0^1 \sin 1/x \, dx$

(g) $$\int_0^1 \frac{1}{\arcsin x} dx = \lim_{\alpha \downarrow 0} \int_\alpha^1 \frac{1}{\arcsin x} dx \underset{\substack{x = \sin y \\ dx = \cos y \, dy}}{=}$$

$$= \lim_{\alpha \downarrow 0} \int_{\arcsin\alpha}^{\pi/2} \frac{\cos y}{y} dy \geq \left(\lim_{\alpha \downarrow 0} \int_{\arcsin\alpha}^{\pi/2} \frac{1}{y} dy\right) - \underbrace{\int_0^{\pi/2} \frac{y}{2} dy}_{=: K < \infty}$$

$$\left[\cos y = \sum_{n=0}^\infty \frac{(-1)^n y^{2n}}{(2n)!} \geq 1 - \frac{y^2}{2} \right.$$

$$= \lim_{\alpha \downarrow 0} \left(\ln y \Big|_{\arcsin\alpha}^{\pi/2} \right) - K = \underbrace{-\lim_{\alpha \downarrow 0} \ln(\underset{\to 0}{\arcsin\alpha})}_{= \infty} + \ln\frac{\pi}{2} - K$$

exist. nicht.