Blatt 2 Hausaufgaben

(H1) Taylorreihen:

(a) $f(x) = \frac{2}{3-4x}$, $T_{n,a}(x) = ?$ für $a = 4$.

$f'(x) = 2(4)\cdot\frac{1}{(3-4x)^2}$, $f''(x) = 2\cdot 4\cdot 4\cdot 2\cdot\frac{1}{(3-4x)^3}$

$f'''(x) = 2\cdot 4\cdot 1\cdot 4\cdot 2\cdot 4\cdot 3\cdot\frac{1}{(3-4x)^4}$

Vermutung: $f^{(k)}(x) = 2\cdot 4^k\cdot k!\,\frac{1}{(3-4x)^{k+1}}$, $k\in\mathbb{N}$

I.A: $k=0$ klar.

I.S. $k\mapsto k+1$: $f^{(k+1)}(x) = \left(f^{(k)}(x)\right)' = 2\cdot 4^k\cdot k!\left(\frac{1}{(3-4x)^{k+1}}\right)' =$

$= 2\cdot 4^k\cdot k!\,(k+1)\cdot 4\,\frac{1}{(3-4x)^{k+2}} = 2\cdot 4^{k+1}(k+1)!\cdot\frac{1}{(3-4x)^{(k+1)+1}}$ ✓

$\Rightarrow T_{n,4}(x) = \sum_{k=0}^{n}\frac{f^{(k)}(4)}{k!}\cdot(x-4)^k = \sum_{k=0}^{n} 2\cdot\frac{4^k\cdot k!}{k!}\,\frac{1}{(3-16)^{k+1}}(x-4)^k$

$= \frac{2}{13}\sum_{k=0}^{n}(-1)^{k+1}\left(\frac{4}{13}\right)^k\cdot(x-4)^k$

(b) $f(x) = \sinh(x)$, $a=0$

$f^{(2k+1)}(x) = \cosh(x)$, $k\in\mathbb{N}_0$

$f^{(2k)}(x) = \sinh(x)$, $k\in\mathbb{N}_0$ $\Rightarrow$ $f^{(2k+1)}(0) = 1$, $f^{(2k)}(0) = 0$, $k\in\mathbb{N}_0$.

$\Rightarrow T_{n,0}(x) = \sum_{k=0}^{n}\frac{f^{(k)}(0)}{k!}x^k = \sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor}\frac{x^{2k+1}}{(2k+1)!}$

(c) $f(x) = \ln(2+x)$, $a=0$

$f'(x) = \frac{1}{2+x}$, $f''(x) = \frac{-1}{(2+x)^2}$, $f'''(x) = \frac{2}{(2+x)^3}$

Induktiv $\Rightarrow$ $f^{(k)}(x) = \frac{(-1)^{k+1}(k-1)!}{(2+x)^k}$, $k\in\mathbb{N}$

$\Rightarrow T_{n,0}(x) = \sum_{k=0}^{n}\frac{f^{k}(0)}{k!}x^k = \sum_{k=1}^{n}\frac{(-1)^{k+1}(k-1)!}{k!}\cdot\frac{1}{2^k}\cdot x^k + \ln 2$

$= \sum_{k=1}^{n}\frac{(-1)^{k+1}}{k\cdot 2^k}\cdot x^k + \ln 2$

(a) $$|R_n(x)| = \left|\frac{f^{(n+1)}(\xi_x)}{(n+1)!}(x-a)^{n+1}\right| = \left|\frac{2\cdot 4^{n+1}\cdot(n+1)!}{(n+1)!}\cdot\frac{1}{(3-4\xi_x)^{n+2}}\cdot(x-4)^{n+1}\right|$$

$$\begin{cases} 2\cdot 4^{n+1}\cdot\left|\frac{(x-4)^{n+1}}{(3-4x)^{n+1}}\right| & , \frac{3}{4}<x<\xi_x<4 \\ 2\cdot\left(\frac{4}{13}\right)^{n+1}\left|(x-4)^{n+1}\right| & , x>\xi_x>4 \end{cases}$$

(b) $$|R_n(x)| = \frac{|f^{(n+1)}(\xi_x)|}{(n+1)!}|x-0|^{n+1} = \begin{cases} \frac{|\sinh(\xi_x)|}{(n+1)!}|x|^{n+1} & n \text{ ungerade} \\ \frac{\cosh(\xi_x)}{(n+1)!}\cdot|x|^{n+1} & n \text{ gerade} \end{cases}$$

$$\le \begin{cases} \frac{|\sinh(x)|}{(n+1)!}\cdot|x|^{n+1} & , n \text{ ungerade} \\ \frac{\cosh(x)}{(n+1)!}\cdot|x|^{n+1} & , n \text{ gerade} \end{cases}$$

(c) $$|R_n(x)| = \left|\frac{f^{(n+1)}(\xi_x)}{(n+1)!}\cdot(x-0)^{n+1}\right| = \frac{n!}{(n+1)!}\cdot\frac{1}{|2+\xi_x|^{n+1}}\cdot|x|^{n+1} \le$$

$$\le \begin{cases} \frac{1}{n+1}\cdot\frac{1}{2^{n+1}}\cdot|x|^{n+1} & , 0<\xi_x<x \\ \frac{1}{n+1}\cdot\frac{|x|^{n+1}}{|2+x|^{n+1}} & -2<x<\xi_x<0 \end{cases}$$

H2 $f(x) = (1+x)^s$, $s \in \mathbb{R}$.

ⓐ $T_{n,a}(x) = ?$ $n \in \mathbb{N}$, $a = 0$.

$f'(x) = s(1+x)^{s-1}$

$f''(x) = s(s-1)(1+x)^{s-2}$

$\vdots$

$f^{(k)}(x) = s(s-1)\cdot\ldots\cdot(s-k+1)\cdot(1+x)^{s-k}$, $k \in \mathbb{N}$

$$\Rightarrow T_{n,0}(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} \cdot x^k = \sum_{k=0}^{n} \underbrace{\frac{s(s-1)\cdot\ldots\cdot(s-k+1)}{k!}}_{\binom{s}{k}} \cdot x^k = \sum_{k=0}^{n} \binom{s}{k} x^k$$

ⓑ Ana I, $\Rightarrow$

Für $s, x \in \mathbb{C}$: $B_s(x) = \sum_{k=0}^{\infty} \binom{s}{k} x^k$ ist verallg. Binomialreihe

Satz 12.13 $\Rightarrow$ Für $s \in \mathbb{R}$, $x \in \mathbb{R}$, $|x|<1$: $B_s(x) = (1+x)^s$ stetig in s.

Satz 15.10 $\Rightarrow$ $B_s(x)$ als Reihe ist schon die Taylorreihe aus ⓐ

H3 Minkowski-Ungl

$f, g \in R[a,b]$

• $p \in [1, \infty)$:

Minkowski-Ungl: $\|f+g\|_p \le \|f\|_p + \|g\|_p$

Zerlege $[a,b]$ in n äquidistante Teilintervalle mit Zerlegungspunkten $x_k := a + k \cdot \frac{b-a}{n}$, $k = 0, \ldots, n$

Nach Satz 13.28 gilt:

$$\|f+g\|_p = \left(\int_a^b |f(x)+g(x)|^p \, dx \right)^{1/p} =$$

$$\underset{\uparrow}{=} \lim_{n\to\infty} \left(\sum_{k=1}^{n} |f(x_k)+g(x_k)|^p \cdot \frac{b-a}{n} \right), \text{ da } |f+g|^p \in R[a,b]$$

Riemann-Summen (13.28)

Andererseits:

$$\left(\sum_{k=1}^{n} |f(x_k)+g(x_k)|^p \cdot \frac{b-a}{n}\right)^{1/p} \underset{\substack{11.34.\\ \text{gewöhnl.}\\ \text{MU}}}{\leq}$$

$$\leq \left[\left(\sum_{k=1}^{n} |f(x_k)|^p\right)^{1/p} + \left(\sum_{k=1}^{n} |g(x_k)|^p\right)^{1/p}\right]\cdot\left(\frac{b-a}{n}\right)^{1/p}$$

$$= \left(\sum_{k=1}^{n} |f(x_k)|^p \cdot \frac{b-a}{n}\right)^{1/p} + \left(\sum_{k=1}^{n} |g(x_k)| \cdot \frac{b-a}{n}\right)^{1/p} \xrightarrow[13.28]{n\to\infty} \|f\|_p + \|g\|_p$$

- $\underline{p=\infty}$: $\|f+g\|_\infty = \sup_{x\in[a,b]} |f(x)+g(x)| \leq \sup_{x\in[a,b]} |f(x)| + \sup_{x\in[a,b]} |g(x)| = \|f\|_\infty + \|g\|_\infty$ □

H4 Sei $x>0$. Zeige $F(y) := \ln\left(\int_0^1 t^{x-1}(1-t)^{y-1}\,dt\right)$ konvex auf $(0,\infty)$.

$x,y>0$: Beta-Fkt $B(x,y) = \frac{\Gamma(x)\cdot\Gamma(y)}{\Gamma(x+y)} = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt$

Zeige: $F(y) = \ln B(x,y)$ konvex auf $(0,\infty)$ in y.

Seien $0<z<y$, $\lambda\in(0,1)$, $p=\frac{1}{\lambda}>1$, $q=\frac{1}{1-\lambda}>1$: $\frac{1}{p}+\frac{1}{q}=1$,

$f(t) := t^{\frac{x-1}{p}}\cdot(1-t)^{\frac{z-1}{p}}$, $g(t) := t^{\frac{x-1}{q}}\cdot(1-t)^{\frac{y-1}{q}}$,

sowie $0<\varepsilon<R<1$.

Hölder-Ungl. 13.30 $\Longrightarrow$

$$\int_\varepsilon^R t^{x-1}(1-t)^{\lambda z+(1-\lambda)y-1}\,dt = \int_\varepsilon^R |f(t)\cdot g(t)|\,dt \overset{\text{Hölder}}{\leq}$$

$$\leq \left(\int_\varepsilon^R |f(t)|^p\,dt\right)^{1/p}\cdot\left(\int_\varepsilon^R |g(t)|^q\,dt\right)^{1/q} = \left(\int_\varepsilon^R t^{x-1}\cdot(1-t)^{z-1}\,dt\right)^{\lambda}\cdot\left(\int_\varepsilon^R t^{x-1}\cdot(1-t)^{y-1}\,dt\right)^{1-\lambda}$$

Für $\varepsilon\downarrow 0$, $R\uparrow 1$ folgt:

$$B(x,\lambda z+(1-\lambda)y) \leq B(x,z)^{\lambda}\cdot B(x,y)^{1-\lambda}$$

$$\Rightarrow F(\lambda z+(1-\lambda)y) = \ln B(x,\lambda z+(1-\lambda)y) \leq \ln\left(B(x,z)^{\lambda}\cdot B(x,y)^{1-\lambda}\right)$$

$$= \lambda\cdot\ln B(x,z) + (1-\lambda)\cdot B(x,y) = \lambda\cdot F(z) + (1-\lambda)F(y)$$

□