Aufgabe 26

$S_r^2 = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 = r^2\}$ $\qquad (r>0)$

$\mu(y) := \int_{S_r^2} \frac{\rho}{\|x-y\|} \, dS \quad , \; y \in \mathbb{R}^3$

Beh.:

$$\mu(y) = \begin{cases} 4\pi r \rho & \text{für } \|y\| \leq r \\ \frac{4\pi r^2 \rho}{\|y\|} & \text{für } \|y\| \geq r \end{cases}$$

Beweis: ① μ ist rotationssymmetrisch

② Wegen ① genügt es, die Behauptung für $y = (0,0,a)$ mit $a > 0$ zu zeigen
($a = 0$ klar)

$$\gamma(u,v) = \begin{pmatrix} r\cos v \cos u \\ r \cos v \sin u \\ r \sin v \end{pmatrix} \quad , \quad (u,v) \in \,]0,2\pi[\times]-\tfrac{\pi}{2}, \tfrac{\pi}{2}[$$

$\gamma : \underbrace{]0,2\pi[\times]-\tfrac{\pi}{2}, \tfrac{\pi}{2}[}_{=:\Omega} \longrightarrow S_r^2 \setminus N$ Einbettung

wobei N 2-dim. Hausdorff-Nullmenge

$$\int_{S_r^2} \frac{\rho}{\|x-y\|} \, dS = \rho \cdot \int_\Omega f(\gamma(u,v)) \cdot \sqrt{g^\gamma(u,v)} \; d\lambda^2(u,v)$$

$$\left(\text{mit } f(x) = \frac{1}{\|x-y\|}\right)$$

$$= \rho \int_\Omega \frac{1}{\sqrt{r^2\cos^2 v \cos^2 u + r^2 \cos^2 v \sin^2 u + (r \sin v - a)^2}} \cdot r^2 \cos v \; d\lambda^2(u,v)$$

$$= \rho \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{0}^{2\pi} \frac{1}{\sqrt{r^2\cos^2 v + r^2 \sin^2 v - 2ar\sin v + a^2}} \cdot r^2 \cos v \, du \, dv$$

$$= 2\pi\rho r^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{\sqrt{r^2 - 2ar\sin v + a^2}} \cos v \, dv$$

$$z = \sin v$$

$$\frac{dz}{dv} = \cos v \iff \cos v \, dv = dz$$

$$= 2\pi\rho r^2 \int_{-1}^{1} \frac{1}{\sqrt{r^2 - 2arz + a^2}} \, dz$$

$$t = r^2 - 2arz + a^2$$

$$\frac{dt}{dz} = -2ar \iff dz = -\frac{1}{2ar} dt$$

$$= 2\pi\rho r^2 \int_{(r+a)^2}^{(r-a)^2} \frac{1}{\sqrt{t}} \left(-\frac{1}{2ar}\right) dt$$

$$= -\frac{\pi\rho r}{a} \int_{(r+a)^2}^{(r-a)^2} t^{-\frac{1}{2}} \, dt$$

$$= -\frac{\pi\rho r}{a} \left[2t^{\frac{1}{2}} \right]_{(r+a)^2}^{(r-a)^2}$$

$$= -\frac{2\pi\rho r}{a} \left(|r-a| - (r+a) \right)$$

$$= \begin{cases} -\frac{2\pi\rho r}{a}(-2a) & \text{falls } a \leq r \\ -\frac{2\pi\rho r}{a}(-2r) & \text{falls } a > r \end{cases}$$

$$= \begin{cases} 4\pi\rho r & \text{falls } a \leq r \\ \frac{4\pi\rho r^2}{a} & \text{falls } a > r \end{cases}$$